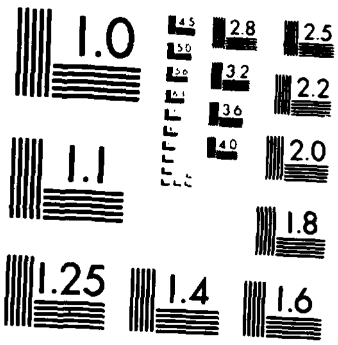
AD-A180 200 A GENERALIZED QUANTILE ESTIMATOR UNDER CENSORING(U) 1/1
SOUTH CAROLINA UNIV COLUMBIA DEPT OF STATISTICS
Y L LIO ET AL. FEB 87 TR-124 AFOSR-TR-87-1617
UNCLASSIFIED AFOSR-84-0156 F/G 12/3 NL



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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release, distribution unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Statistics Technical Report No. 124	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 87-1617
6a. NAME OF PERFORMING ORGANIZATION Department of Statistics / 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code) University of South Carolina Columbia, SC 29208	7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB, DC 20332
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR, ARO / 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-84-0156
8c. ADDRESS (City, State and ZIP Code) Bolling AFB, DC 20332	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. A5; WORK UNIT NO.
11. TITLE (Include Security Classification) A Generalized Quantile Estimator under Censoring	
12. PERSONAL AUTHOR(S) Y. L. Lio and W. J. Padgett	
13a. TYPE OF REPORT: Technical / 13b. TIME COVERED FROM ____ TO ____	14. DATE OF REPORT (Yr., Mo., Day): February, 1987 / 15. PAGE COUNT: 20
16. SUPPLEMENTARY NOTATION	
17. COSATI CODES: FIELD / GROUP / SUB. GR.	18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) Random Censoring; Product-Limit quantile estimator; Generalized Product-Limit quantile estimator; Median survival time estimation; Nonparametric quantile estimation.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL Maj. Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code) (202) 765-5027 / 22c. OFFICE SYMBOL NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE. UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

AFOSR-TR- 87-1617

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Y. L. Lio and W. J. Padgett

University of South Carolina
Statistics Technical Report No. 124
62G05-18

February, 1987

Department of Statistics
University of South Carolina
Columbia, South Carolina 29208

* Research supported by the United States Air Force Office of Scientific Research Grant AFOSR-84-0156 and Army Research Office Grant MIPR ARO 139-85.

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Y. L. Lio and W. J. Padgett

Department of Statistics
University of South Carolina
Columbia, SC 29208

ABSTRACT

Based on right-censored data from a lifetime distribution F_0, a smooth alternative to the product-limit estimator as a nonparametric quantile estimator of a population quantile is proposed. The estimator is a "generalized product-limit quantile" obtained by averaging appropriate subsample product-limit quantiles over all subsamples of a fixed size. Under the random censorship model and some conditions on F_0, it is shown that the estimator is consistent and has the same asymptotic normal distribution as the product-limit quantile estimator. A small Monte Carlo simulation study shows that there exist some values of the subsample size for which the estimator performs better than the product-limit quantile estimator in the sense of estimated mean squared errors.

1. INTRODUCTION

Arbitrarily right-censored data arise naturally in industrial life testing and medical follow-up studies. In these situations it is important to be able to obtain nonparametric estimates of various characteristics of the survival function S. One characteristic of the survival distribution that is of interest is the quantile function. For any probability distribution function G, the quantile function is defined by $Q(p)=G^{-1}(p)=\xi_p = \inf\{x: G(x) \geq p\}$, $0 \leq p \leq 1$.

For a random (uncensored) sample $X_1,\ldots,X_n$ from G, the sample quantile function $G_n^{-1}(p)=\inf\{ x: G_n(x) \geq p\}$, $0\leq p \leq 1$, has been used to estimate ξ_p, where $G_n(x)$ denotes the empirical distribution function. Csörgő (1983) gave many of the known results concerning $G_n^{-1}(p)$. Kaigh and Lachenbruch (1982) considered a "generalized sample quantile" obtained by averaging an appropriate subsample quantile over all subsamples of a given size.

For arbitrarily right-censored data, Sander (1975) proposed estimation of ξ_p by the quantile function of the product-limit (PL) estimator of S. She and Cheng (1984) obtained some asymptotic properties of that estimator. Reid (1981) studied influence functions for any Fréchet-differentiable function of the PL estimator, gave the influence function of the PL quantile function, and obtained the same asymptotic normality for the PL quantile function as Cheng (1984) did. Padgett (1986) proposed a kernel-type estimator which smoothed the PL quantile function.

The intent of this paper is to propose and study a <u>generalized</u> PL <u>quantile</u> <u>estimator</u> based on right-censored data. This generalized PL quantile estimator is obtained by averaging subsample PL quantile estimates over all subsamples with subsample size k from a right-censored sample of size n, where $1\leq k\leq n$. When censoring is not present, this generalized PL quantile estimator becomes the generalized quantile estimator proposed by Kaigh and Lachenbruch (1982).

It will be shown that under some nonrestrictive conditions the generalized PL quantile estimator satisfies the properties of U-statistics, and the asymptotic normality for the estimator is presented in Section 3. It should be mentioned that the order statistic methods used by Kaigh and Lachenbruch (1982) to obtain an expression for the asymptotic variance of the generalized sample quantile function for uncensored data cannot be used in the case of right-censored observations. This is due to the unequal random jumps in the PL distribution function. Based on the results of a small Monte Carlo simulation study reported in Section 4, in many cases there is a subsample size k for which the generalized PL quantile estimator performs better than the PL quantile estimator in the sense of smaller estimated mean squared errors.

For the generalized PL quantile estimator, one problem is the optimal choice (in some sense) of the subsample size k. Since no results on the exact mean squared error of the proposed estimator are currently available, the subsample size that minimizes the mean squared error cannot be obtained. Bootstrap methods for randomly right-censored data (Efron 1981) might be used in some cases, however, to estimate the optimal subsample size from the data but would require extensive amounts of computer time. This procedure is still under study.

2. GENERALIZED PL QUANTILE ESTIMATOR

Let $X_1^0,\ldots,X_n^0$ be the true survival times of n items or individuals that are censored on the right by a sequence $U_1,\ldots,U_n$, which are independent random variables with identical distribution H (usually unknown). It is assumed that the X_i^0's are nonnegative independent identically distributed random variables with common unknown distribution function F_0 and unknown quantile function $Q^0(p)=\xi_p^0=\inf\{t: F_0(t)\geq p\}$, $0\leq p\leq 1$.

The observed right-censored data are denoted by the pairs (X_i, Δ_i), $i=1,\ldots,n$, where

$$X_i = \min\{X_i^0, U_i\}, \qquad \Delta_i = 1 \text{ if } X_i^0 \leq U_i$$
$$= 0 \text{ if } X_i^0 > U_i.$$

Thus, it is known which observations are times of failure or death and which ones are censored or loss times. For this model the distribution function of X_i is $F=1-(1-F_0)(1-H)$.

Based on the censored sample (X_i, Δ_i), $i=1,2,\ldots,n$, a popular estimator of the survival function $1-F_0(t)$ at $t\geq 0$ is the PL estimator, proposed by Kaplan and Meier (1958) as the "nonparametric maximum likelihood estimator." Efron (1967) showed that this estimator is "self-consistent." Let (Z_i, Λ_i), $i=1,\ldots,n$, denote the ordered X_i's along with their corresponding Δ_i's. Then the PL estimator of $1-F_0(t)$ is defined by

$$\hat{P}_n(t) = \begin{cases} 1, & 0 \le t \le Z_1, \\ \prod_{i=1}^{k-1} \left(\frac{n-i}{n-i+1}\right)^{\Lambda_i}, & Z_{k-1} < t \le Z_k,\ k=2,\ldots,n \\ 0, & Z_n < t. \end{cases}$$

Denote the PL estimator of $F_0(t)$ by $\hat{F}_n = 1-\hat{P}_n$.

Based on randomly right-censored data, it is natural to estimate the quantile function $Q^0(p)$ by the PL quantile function $\hat{Q}_n(X_1,\Delta_1,\ldots,X_n,\Delta_n;p) \equiv \hat{Q}_n(p) = \inf\{t: \hat{F}_n(t) \ge p\}$. Cheng (1984) obtained asymptotic normality results for $\hat{Q}_n$. Reid (1981) got the influence curve for $\hat{Q}_n$.

A generalized PL quantile estimator of $Q^0(p)$, $0 \le p \le 1$, based on the randomly right-censored observations (X_i, Δ_i), $i=1,\ldots,n$ is defined as follows:

For a fixed integer k, $1 \le k \le n$, consider the selection of a simple random sample (X_{ki}, Δ_{ki}), $i=1,\ldots,k$ (without replacement) from the right-censored data (X_i, Δ_i), $i=1,\ldots,n$. By using this subsample (X_{ki}, Δ_{ki}), $i=1,\ldots,k$, the PL pth quantile estimator $\hat{Q}_k(X_{k1}, \Delta_{k1}, \ldots, X_{kk}, \Delta_{kk}; p) \equiv \hat{Q}_k(p)$, $0 \le p \le 1$, is obtained as defined above. Then the generalized PL quantile $K_{p;k;n}$, $0 \le p \le 1$, is defined to be the average of the subsample PL pth quantile estimators over all $\binom{n}{k}$ subsamples of size k from (X_i, Δ_i), $i=1,\ldots,n$. Therefore

$$K_{p;k;n} = \frac{1}{\binom{n}{k}} \sum_{C_n} \hat{Q}_k(p), \tag{2.1}$$

where C_n indicates that the summation is over all combinations $\{k1,\ldots,kk\}$ of k integers selected from $\{1,\ldots,n\}$. Subject to the obvious constraint $1 \le k \le n$, the assumed subsample size is arbitrary and the choice k=n in (2.1) gives $K_{p;k;n} = \hat{Q}_n$. Thus, the statistics defined by (2.1) form a collection of "generalized quantile estimators" which includes the usual PL quantile estimator. This estimator (2.1) is obviously a U-statistic (Hoeffding, 1984) with kernel $\dot{Q}_k$ which is symmetric with respect to $(X_1,\Delta_1),\ldots,(X_k,\Delta_k)$.

A PL quantile $\dot{Q}_k$ is not in general an unbiased estimator of the corresponding quantile of the lifetime distribution, although Lio and Padgett (1986) have shown that any bias becomes negligible with increasing sample size k. Appealing to a monotonicity principle would suggest that the subsampling scheme provides an estimator $K_{p;k;n}$ of ξ_p with bias magnitude exceeding that of the PL quantile estimator $Q_n(X_1,\Delta_1,\ldots,X_n,\Delta_n;p)$. However, it would seem plausible also that the averaging procedure might result in a reduction of sampling variability which is adequate to decrease the mean squared error of estimation. This property will be indicated in the Monte Carlo simulations reported in Section 4.

Due to the censoring, $K_{p;k;n}$ is not a simple linear combination of order statistics of lifetime data. When $\Delta_i=1$, $i=1,..,n$ (i.e. no observation is censored), the generalized PL quantile estimator reduces to the estimator proposed by Kaigh and Lachenbruch (1982) which is a linear combination of order statistics of the random sample. They then used order statistic properties to get the asymptotic variance and mean squared error for their generalized quantile estimator. In the case of random right-censoring, similar results for the variance, bias, and mean squared error of the generalized PL quantile estimator and the mean squared consistency of the generalized PL quantile estimator seem to be difficult to obtain under general conditions on F_0 and H. Some asymptotic results, however, have been obtained under reasonable conditions and are discussed in the next section.

3. SOME ASYMPTOTIC RESULTS

In this section, asymptotic normality for the generalized PL quantile estimator is presented. The somewhat lengthy proof of Theorem 3.2 will be given in the Appendix.

For a distribution function G, let $T_G \equiv \sup\{t: G(t)<1\}$.

Theorem 3.1. Let p be such that $0 \leq p < \min\{1, T_{H(Q^0)}\}$. Suppose H is continuous and Q^0 is differentiable in a neighborhood of p with bounded first derivative on a neighborhood of p. If for fixed $1 \leq k \leq n$, $E((\hat{Q}_k(p))^2) < \infty$, then $n^{1/2}(K_{p;k;n} - u_{p;n}) \longrightarrow Z$ in distribution as $n \to \infty$, where $u_{p;n} = E(K_{p;k;n})$ and Z is a normal random variable with mean zero and variance $\sigma^2_{p;k} = k^2 \mathrm{var}(E(\hat{Q}_k(p) | X_1))$.

The proof of Theorem 3.1 follows from U-statistic properties of Hoeffding (1948).

Due to censoring, a simple expression for the asymptotic variance of this generalized PL quantile estimator $K_{p;k;n}$ is difficult to get, and the bias term is also unknown. Therefore, theoretical comparison of this generalized PL quantile and PL quantile estimators in terms of asymptotic variance or in terms of mean squared error is still not available. However, simulation results presented in the Section 4 indicate a range of possible values of subsample size k for which the mean squared errors of $K_{p;k;n}$ are less than those of the PL quantile estimator for each p.

By U-statistic properties (Hoeffding, 1948), it is easy to show for fixed $1 \leq k \leq n$, n $\mathrm{var}(K_{p;k;n})$ decreases to lower limit $\sigma^2_{p;k}$. In fact, an immediate application of Theorems 5.1 and 5.2 of Hoeffding (1948), gives the following corollary.

<u>Corollary 3.1</u>. For $0 \leq p < \min\{1, T_{H(Q^0)}\}$ and fixed $1 \leq k \leq n$, we have $\sigma^2_{p;k} \leq n\, \mathrm{var}(K_{p;k;n}) \leq k\, \mathrm{var} Q_k(p)$.

In Theorem 3.1 and Corollary 3.1 the existence of second moments of the PL quantile estimator was assumed. Lio and Padgett (1986) has proven that for sufficiently large k, $E[(\hat{Q}_k(p))^2]$ is finite and the bias becomes negligible. Therefore, we develop a companion result to Theorem 3.1 which provides the asymptotic normality of the generalized PL quantile estimator as the subsample size also increases to infinity.

<u>Theorem 3.2</u>. Let p be such that $0 \leq p < \min\{1, T_{H(Q^0)}\}$. Suppose H and F_0 are continuous and Q^0 is differentiable in a neighborhood

of p with bounded first derivative on a neighborhood of p. If $k_n \to \infty$ and $\lim\inf(k_n/n) = c$ (> 0) as $n \to \infty$, where c is some constant, then

$$n^{1/2}(K_{p;k_n;n} - Q^0(p)) \to Z \text{ in distribution,}$$

where Z is a normal random variable with mean zero and variance

$$\sigma_p^2 = (Q^{0\prime}(p))^2 (1-p)^2 \int_0^p \frac{dx}{(1-x)^2(1-H(Q^0(x)))} .$$

Note that the limiting distribution in Theorem 3.2 is the same as that for $\hat{Q}_n(p)$ obtained by Cheng (1984).

4. SOME SIMULATION RESULTS

Often only small samples are available in real situations due to the expense or difficulties to obtain lifetime data. Hence, some investigation of the small sample properties of $K_{p;k;n}$ is needed. Therefore, a small Monte Carlo simulation study was performed for two common families of lifetime distributions. These distributions are the exponential distribution with density $f(x)= \beta \exp(-\beta x)$, $x>0$, $\beta=1$, and the Weibull distribution with density

$$f(x)= \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(- \left(\frac{x}{\beta} \right)^{\alpha} \right), \quad x > 0,$$

$(\alpha,\beta)=(0.5,1)$, $(2,1)$, $(2,5)$. Two censoring distributions H were used: exponential with density $h(u)=\lambda e^{-\lambda u}$, $u>0$, $\lambda>0$, and uniform on the interval $(0, \lambda)$, $\lambda>0$.

The parameter λ of the censoring distribution was determined to give either 30% or 50% censoring. That is, λ was determined so that the probability of a censored observation, $pr(X^0 > U)=0.3$ or 0.5, at least approximately. This probability was calculated by numerical integration using the midpoint rule when it could not be obtained exactly. The value of λ is reported in the resulting table for each case.

Since the generalized PL quantile estimator is not a linear combination of order statistics of lifetime data and it can not be simplified, the simulation procedure was the following: For given sample size n, let $1 \le k \le n$.

Step 1. A random sample of size n, $X_1^0,\ldots,X_n^0$, was generated from the lifetime distribution.

Step 2. A random sample $U_1,\ldots,U_n$ was generated from the censoring distribution.

Step 3. The censored sample $(X_1,\Delta_1),\ldots,(X_n,\Delta_n)$ was obtained by

$X_i = \min(X_i^0,U_i)$

$$\Delta_i = \begin{cases} 1 & \text{if } X_i = X_i^0 \\ 0 & \text{if } X_i = U_i, \end{cases} \quad i=1,\ldots,n.$$

Step 4. Take a random sample of pairs (X_{ki},Δ_{ki}), $i=1,\ldots,k$, (without replacement) from (X_i,Δ_i), $i=1,\ldots,n$.

Step 5. Order the X_{ki}'s, $i=1,\ldots,k$, from the smallest to the largest.

Step 6. Compute the PL estimator of lifetime distribution based on the chosen subsample of size k.

Step 7. For each value p=0.10, 0.25, 0.50, 0.75, 0.90, 0.95, compute the PL quantile estimator based on the chosen subsample.

The above procedure (from step 4 to step 7) was repeated for all $\binom{n}{k}$ different subsamples.

Step 8. Average all $\binom{n}{k}$ subsample results to get $K_{p;k;n}$.

Repeat step 1 through step 8 for N samples.

Step 9. The mean squared error, sample variance, and bias of $K_{p;k;n}$ was computed over all N samples.

Step 10. For each $1\leq k\leq n$ and each p, the ratios of mean squared error for PL quantile estimator and mean squared error for $K_{p;k;n}$ were calculated.

For given sample size n, this simulation procedure needs $2^n - 1$ loops for each sample, therefore a large amount of cpu time was required for each simulation. For example: When N=200, n=20, the required cpu time is over 10 days on a DEC VAX 11-750 computer. When n=30, the cpu time is over 30 days on a DEC VAX 11-750 computer and hence is impractical. It seems that cpu time increases with exponential rate when sample size n increases. Therefore, sample sizes of n=10, 20 were chosen in the simulation study, and for each case simulated (i.e. each distribution, p, and sample size combination), 200 censored samples were generated using the random number generator in the International Mathematical and Statistical Library (IMSL, 1982) on an FPS 264 attached processor to an IBM 3081 computer. For each simulation case using $n=20$, $N=200$, the cpu time was 24.75 seconds on that system.

Some of the results of the simulations are shown in Tables 4.1-4.12 which contain the ratios of estimated mean squared error for PL quantile estimator and estimated mean squared error for the generalized PL quantile estimator, $MSE(\dot{Q}_n(p))/MSE(K_{p;k;n})$.

In each case, for $p<0.90$, there is a k value for which $K_{p;k;n}$ has smaller estimated mean squared error than that of the PL quantile estimator. In particular, this is true for several k values for the median estimators $K_{0.5;k;n}$. For small p and large p, those k values such that $K_{p;k;n}$ has smaller mean squared error than the PL quantile estimator are close to the sample size n. In all cases, the subsample size k giving the largest ratio of estimated mean squared error tends to decrease with p up to about 0.5, and then increase for larger p. The parameter k determines the amount of smoothing of this estimator, small k indicating more smoothing. Therefore, more smoothing is needed in the middle of the distribution than in the tails to decrease the mean squared error of the generalized PL quantile estimator.

Increasing the amount of censoring from 30% to 50% seems to have little effect on the estimated ratio of mean squared errors, especially for large values of p. Also, the behavior of the estimator is similar for the two censoring distributions used in the simulations.

5. AN EXAMPLE

As an example of the generalized PL quantile estimator, 15 observations were chosen from the lung cancer data (standard, squamous) in Data Set I of Kalbfleisch and Prentice (1980, p. 223). The data, transformed to "months," are shown in Table 5.1, where "+" denotes a censored observation. So, $\Lambda_4=\Lambda_8=0$ and all other Λ_i's are one. The quantile estimators $K_{p;k;n}$ computed from this data with k=8,10,13 are shown in Figures 5.1-5.3 along with the PL quantile, showing the smoothing that has been obtained.

TABLE 5.1 EXAMPLE DATA

i	1	2	3	4	5	6	7	8
Z_i	0.27	0.33	0.37	0.83^+	1.40	2.40	2.73	3.33^+
i	9	10	11	12	13	14	15	
Z_i	3.67	3.93	4.20	4.80	7.60	10.47	13.70	

APPENDIX

In order to prove Theorem 3.2, the representation of Peterson (1977) for the Kaplan-Meier estimate $\hat{F}_n(t)$ of the lifetime distribution is used. The PL p-th quantile estimate $\hat{Q}_k(p)$ based on sample (X_i, Δ_i), $i=1,\ldots,k$, is considered as a function $V(S_k^u, S_k^c, p)$ of two empirical subsurvival functions S_k^u, S_k^c, where

$$1-S_k^u(t) = \frac{1}{k}\sum_{i=1}^{k} I[X_i \le t,\ \delta_i = 1],$$

$$1-S_k^c(t) = \frac{1}{k}\sum_{i=1}^{k} I[X_i \le t,\ \delta_i = 0].$$

The corresponding p-th quantile is the value of V at (S^u, S^c) where $S^u(t)=P[X_i>t,\ \delta_i=1]$ and $S^c(t)=P[X_i>t,\ \delta_i=0]$. Let β be the space of subsurvival functions, i.e. the space of decreasing left continuous functions from R into $[0,\alpha]$, where $\alpha \le 1$. Let $||W_1-S_1||_\infty = \sup_{0\le x<\infty} | W_1(x)-S_1(x) |$ where W_1 and S_1 belong to β. Let β^2 be the product space of β and itself, and define a norm on β^2 such that for any W_1, $W_2 \in \beta^2$ $||W_1 - W_2||^2 = ||W_{11}-W_{21}||_\infty^2 + ||W_{12}-W_{22}||_\infty^2$, where W_{ij} is the jth component of W_i, j=1,2, i=1,2. Clearly, $||W_1 - W_2|| \le ||W_{11}-W_{21}||_\infty + ||W_{12}-W_{22}||_\infty$. Reid (1981) proved that V is Fréchet-differentiable at S^u and S^c with respect to $||\cdot||_\infty$ in each argument S^u and S^c for $0\le p< \min\{1, T_{H(Q^0)}\}$, and has continuous partial derivatives. Therefore, V is Fréchet-differentiable at (S^u, S^c) with repect to $||\cdot||$ for

$0 \leq p < \min\{1, T_{H(Q^0)}\}$, and

$$V(S^u_k, S^c_k, p) - V(S^u, S^c, p)$$

$$= \frac{-1}{k} \sum_{i=1}^{k} IC_1(V, S^u, S^c; X_i)(p)\, I_{[\delta_i=1]}$$

$$- \frac{1}{k} \sum_{i=1}^{k} IC_2(V, S^u, S^c; X_i)(p)\, I_{[\delta_i=0]}$$

$$- \int IC_1(V, S^u, S^c; s)(p)\, dS^u(s) - \int IC_2(V, S^u, S^c; s)(p)\, dS^c(s)$$

$$+ o(||S_k - S||), \tag{A.1}$$

where $S_k = (S^u_k, S^c_k)$ and $S = (S^u, S^c)$. Now,

$$\frac{1}{\binom{n}{k_n}} \sum \hat{Q}_{k_n}(p) - V(S^u, S^c, p)$$

$$= \frac{1}{\binom{n}{k_n}} \sum [\, V(S^u_{k_n}, S^c_{k_n}, p) - V(S^u, S^c, p)]$$

$$= \frac{1}{\binom{n}{k_n}} \sum_{C} \{ \frac{-1}{k_n} \sum_{i=1}^{k_n} IC_1(V, S^u, S^c; X_i)(p)\, I_{[\delta_i=1]}$$

$$- \frac{1}{k_n} \sum_{i=1}^{k_n} IC_2(V, S^u, S^c; X_i)(p)\}\, I_{[\delta_i=0]}\}$$

$$+ \Big(\frac{1}{\binom{n}{k_n}} \sum ||S_{k_n} - S|| \Big)\, o(1)$$

$$= U_n + \Big(\frac{1}{\binom{n}{k_n}} \sum ||S_{k_n} - S|| \Big)\, o(1),$$

where IC_1 and IC_2 are influence curves of the PL quantile function (see Reid, 1981).

We need the following lemma to complete the proof.

<u>Lemma 1</u>. $n^{1/2}\, U_n \longrightarrow Z$ in distribution, where Z is a normal random variable with mean zero and variance σ^2_p.

<u>Proof</u>. U_n is a U-statistic with

$$h(X_1, \ldots, X_{k_n}) = \frac{-1}{k_n} \sum_{i=1}^{k_n} IC_1\, I_{[\delta_i=1]} - \frac{1}{k_n} \sum_{i=1}^{k_n} IC_2\, I_{[\delta_i=0]}$$

and $E(h)=0$ (see example 2 and p.83 of Reid, 1981).

It is easy to show that $Var(U_n) = \frac{1}{\binom{n}{k_n}} \sum_{c=1}^{k_n} \binom{k_n}{c} \binom{n-k_n}{k_n-c} \zeta_c$,

where for $1 \le c \le k_n$

$$\zeta_c = E[h(X_1,\dots,X_c,X_{c+1},\dots,X_{k_n}) * h(X_1,\dots,X_c,X_{k_n+1},\dots,X_{2k_n-c})]$$

$$= \frac{c}{k_n^2} \left[\int IC_1^2 \; dF^u(s) + \int IC_2^2 \; dF^c(s) \right]$$

$$= \frac{c}{k_n^2} \sigma_p^2.$$

Therefore, $Var(U_n) = \frac{1}{n} \sigma_p^2$.

Let $V_n^* = \frac{k_n}{n} \sum_{i=1}^{n} \{ h_1(X_i) \}$

$$= -\frac{1}{n} \left\{ \sum_{i=1}^{n} IC_1 \, I_{[\delta_i=1]} + \sum_{i=1}^{n} IC_2 \, I_{[\delta_i=0]} \right\}$$

where $h_1(x) = E[h(x,X_2,\dots,X_{k_n})]$. Since $E[h^2] < \infty$ (Reid, 1981),

using the same argument as the proof of Theorem 3.3.13 of Randles and Wolfe (1979), it is easy to get

$$n\,E[(U_n-V_n^*)^2] = nE[(U_n)^2] + nE[V_n^{*\,2}] - 2nE[U_n V_n^*]$$

$$= n\,Var(U_n) + \sigma_p^2 - 2\,k_n^2\,\zeta_1 = 0.$$

Reid (1981) has proven that $n^{1/2} V_n^* \longrightarrow Z$ in distribution.

Therefore $n^{1/2} U_n \longrightarrow Z$ in distribution. //

Since $\| S_{k_n} - S \| \le \| S_{k_n}^c - S^c \|_\infty + \| S_{k_n}^u - S^u \|_\infty$

and $k_n^{1/2} \| S_{k_n}^c - S^c \|_\infty = O_p(1)$, $k_n^{1/2} \| S_{k_n}^u - S^u \|_\infty = O_p(1)$, by using a basic argument the following Lemma is easy to prove.

<u>Lemma 2</u>. $\frac{k_n^{1/2}}{\binom{n}{k_n}} [\Sigma \, \| S_{k_n} - S \|] = O_p(1).$

With Lemma 1 and Lemma 2 and (A.1), letting $k_n \longrightarrow \infty$ and $\lim\inf(k_n/n) = c$ (>0) for some constant c as $n \longrightarrow \infty$, Theorem 3.2 is proven. ///

TABLE 4.1 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: E(1), CENSORING DISTRIBUTION: E(3/7)
n=20 (30% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.02	0.04	0.08	0.17	0.35	0.64	1.09	1.08	1.50	0.54	0.59	0.56	0.73	0.89	1.23	1.50	1.45	1.30	1.68
0.25	0.13	0.26	0.84	2.19	0.69	0.88	1.17	1.20	1.19	0.89	1.13	1.27	1.28	1.24	1.50	1.90	1.51	1.46	1.08
0.50	3.75	1.82	2.00	2.46	1.21	1.52	1.15	0.93	1.28	1.08	0.99	1.35	1.16	0.94	1.20	1.23	1.19	1.34	1.13
0.75	0.96	2.48	5.00	4.86	3.82	3.81	3.73	2.57	3.10	2.71	2.26	2.99	1.90	1.47	2.04	1.86	1.83	2.45	1.83
0.90	0.29	0.44	0.65	0.92	1.12	1.38	1.51	1.69	1.82	1.92	1.81	1.82	1.51	1.16	1.39	1.40	1.17	0.84	1.30
0.95	0.20	0.26	0.34	0.42	0.50	0.57	0.61	0.75	0.76	0.85	0.90	0.86	1.09	1.00	0.98	0.93	1.06	0.79	0.98

TABLE 4.2 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: E(1), CENSORING DISTRIBUTION: E(1)
n=20 (50% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.06	0.06	0.09	0.15	0.27	0.43	0.79	0.74	1.33	0.65	0.56	0.59	0.72	0.87	1.07	1.38	1.45	1.31	1.56
0.25	0.60	0.58	0.98	2.09	0.85	0.78	1.02	0.86	1.05	0.93	0.68	1.15	1.14	1.30	1.44	1.73	1.17	1.53	1.38
0.50	3.18	2.71	6.28	6.60	3.45	2.67	2.41	1.87	1.78	1.83	1.44	2.03	1.07	1.07	1.23	1.32	1.55	1.34	1.06
0.75	0.46	0.80	1.43	1.61	2.11	2.50	2.85	2.58	2.86	2.61	2.20	2.67	1.73	1.29	1.76	1.27	1.76	1.40	1.39
0.90	0.21	0.27	0.35	0.39	0.47	0.55	0.61	0.69	0.73	0.81	0.78	0.81	0.93	0.97	0.96	0.84	1.09	0.99	0.94
0.95	0.28	0.33	0.40	0.43	0.49	0.54	0.58	0.64	0.65	0.70	0.70	0.71	0.81	0.87	0.82	0.78	0.91	0.91	0.86

TABLE 4.3 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: E(1), CENSORING DISTRIBUTION: U(0, 3.1941)
n=20 (approx. 30% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.02	0.04	0.09	0.15	0.36	0.68	1.19	1.20	1.72	0.54	0.56	0.64	0.73	0.85	1.24	1.54	1.38	1.34	1.84
0.25	0.13	0.28	1.15	2.31	0.65	0.90	1.35	1.37	1.07	0.87	1.23	1.31	1.17	1.38	1.49	1.84	1.31	1.42	1.36
0.50	3.82	1.76	2.10	2.38	1.41	1.85	1.21	0.98	1.15	0.79	0.99	0.96	1.10	1.18	1.70	1.61	1.08	1.19	1.30
0.75	0.46	1.39	2.76	2.10	2.69	2.72	2.50	1.79	1.90	1.42	1.49	1.37	1.22	1.21	1.36	1.31	1.03	1.29	0.90
0.90	0.13	0.21	0.27	0.40	0.46	0.56	0.71	0.79	1.02	0.90	1.03	1.10	1.08	1.02	1.21	1.06	1.15	1.26	1.02
0.95	0.16	0.22	0.26	0.33	0.36	0.41	0.47	0.51	0.60	0.59	0.64	0.72	0.73	0.74	0.82	0.81	0.94	0.98	0.90

TABLE 4.4 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: E(1), CENSORING DISTRIBUTION: U(0, 1.5896)
n=20 (approx. 50% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.07	0.08	0.14	0.23	0.38	0.66	1.22	1.18	1.85	0.92	0.79	0.82	1.02	1.14	1.46	1.94	1.83	1.60	2.44
0.25	0.72	0.78	1.97	3.10	1.08	1.15	1.41	1.39	1.38	1.22	1.46	1.59	1.79	1.90	1.69	2.14	1.74	1.98	1.79
0.50	2.04	1.59	5.32	4.12	3.41	3.31	2.47	2.02	1.90	1.67	1.78	1.50	1.44	1.60	1.84	1.73	1.28	1.11	1.31
0.75	0.11	0.20	0.29	0.34	0.42	0.50	0.63	0.58	0.76	0.73	0.80	0.84	0.98	0.86	1.04	1.05	1.23	1.17	0.95
0.90	0.29	0.39	0.45	0.52	0.56	0.61	0.67	0.68	0.75	0.74	0.77	0.81	0.86	0.86	0.89	0.91	0.95	1.01	0.96
0.95	0.44	0.54	0.59	0.66	0.69	0.73	0.77	0.79	0.83	0.83	0.85	0.88	0.91	0.91	0.93	0.95	0.97	1.00	0.97

TABLE 4.5 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: W(2, 1), CENSORING DISTRIBUTION: E(0.425)
n=20 (approx. 30% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.11	0.13	0.23	0.40	0.77	1.15	1.58	1.49	1.55	1.39	1.26	0.98	1.07	1.11	1.38	1.33	1.38	1.21	1.36
0.25	0.46	0.63	1.43	1.66	1.76	1.47	1.46	1.25	1.56	1.22	1.25	1.37	1.51	1.37	1.58	1.41	1.37	1.40	1.25
0.50	0.96	0.62	1.92	1.93	1.78	1.39	1.46	1.14	1.33	1.28	1.18	1.32	1.05	1.06	1.27	1.24	1.08	1.09	0.93
0.75	0.23	0.72	2.03	2.43	2.05	1.88	2.09	1.65	2.05	1.85	1.76	2.07	1.52	1.22	1.58	1.46	1.50	1.50	1.20
0.90	0.13	0.25	0.43	0.71	0.89	1.18	1.33	1.48	1.69	1.66	1.58	1.84	1.39	1.14	1.40	1.26	1.19	1.07	1.10
0.95	0.09	0.15	0.23	0.33	0.40	0.50	0.55	0.71	0.77	0.86	0.94	0.97	1.16	0.97	0.98	0.94	1.08	0.89	0.95

TABLE 4.6 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: W(2, 1), CENSORING DISTRIBUTION: E(0.865)
n=20 (approx. 50% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.32	0.21	0.25	0.35	0.54	0.87	1.22	1.24	1.44	1.59	1.65	1.26	1.28	1.11	1.43	1.35	1.30	1.31	1.21
0.25	2.50	1.15	1.32	1.74	1.64	1.59	1.49	1.10	1.30	1.31	1.15	1.26	1.25	1.23	1.61	1.60	1.10	1.35	1.02
0.50	0.42	0.62	1.58	2.14	1.88	1.75	1.69	1.23	1.18	1.52	1.12	1.33	0.87	0.97	1.24	1.22	1.01	1.02	0.79
0.75	0.19	0.41	0.89	1.26	1.60	1.89	2.12	2.05	2.03	2.20	1.96	1.85	1.37	1.18	1.49	1.28	1.49	1.31	1.27
0.90	0.11	0.18	0.28	0.36	0.47	0.61	0.71	0.86	0.92	1.06	1.08	1.03	1.20	1.04	1.18	0.99	1.30	1.08	1.01
0.95	0.11	0.16	0.23	0.28	0.34	0.41	0.46	0.55	0.57	0.64	0.67	0.65	0.86	0.87	0.82	0.73	1.00	0.93	0.81

TABLE 4.7 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: W(0.5, 1), CENSORING DISTRIBUTION: E(0.375)
n=20 (30% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.00	0.00	0.01	0.03	0.09	0.22	0.48	0.48	1.16	0.15	0.22	0.30	0.50	0.62	1.18	1.65	1.67	1.45	2.61
0.25	0.03	0.09	0.30	1.14	0.19	0.42	0.83	0.89	0.59	0.60	0.83	1.08	0.94	1.31	2.17	2.05	1.45	2.05	2.21
0.50	1.50	5.15	0.81	1.95	0.59	1.27	0.62	0.79	0.77	0.61	0.65	1.11	0.71	0.58	0.68	0.81	1.07	1.32	1.13
0.75	3.11	6.71	10.5	8.31	6.89	6.59	5.32	4.32	4.32	3.68	2.35	3.70	1.77	1.97	1.61	2.29	1.84	2.20	1.68
0.90	0.30	0.37	0.46	0.53	0.62	0.71	0.77	0.89	0.94	0.96	0.92	0.94	1.09	1.04	0.98	0.95	1.20	1.03	0.99
0.95	0.37	0.41	0.47	0.50	0.55	0.59	0.61	0.67	0.68	0.72	0.72	0.72	0.83	0.86	0.80	0.78	0.88	0.90	0.83

TABLE 4.8 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: W(0.5, 1), CENSORING DISTRIBUTION: E(1.33)
n=20 (approx. 50% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0.10	0.01	0.01	0.02	0.04	0.08	0.14	0.31	0.23	0.67	0.15	0.15	0.20	0.35	0.51	1.10	1.32	1.40	1.55	2.60
0.25	0.30	0.38	0.69	1.96	0.44	0.59	0.85	0.69	0.68	0.62	0.55	0.91	0.75	1.07	1.61	2.14	1.30	2.32	0.95
0.50	19.98	11.9	11.8	16.5	5.53	6.91	3.54	3.00	2.32	2.76	1.79	2.68	1.96	1.64	1.50	1.73	1.57	1.23	1.27
0.75	0.27	0.36	0.48	0.48	0.61	0.68	0.78	0.87	0.93	1.02	0.96	0.95	1.03	1.11	1.07	0.85	1.13	0.92	0.86
0.90	0.54	0.59	0.64	0.66	0.70	0.73	0.76	0.80	0.82	0.83	0.85	0.86	0.89	0.93	0.90	0.89	0.95	0.96	0.96
0.95	0.71	0.74	0.78	0.79	0.82	0.84	0.86	0.88	0.89	0.90	0.91	0.92	0.93	0.96	0.94	0.94	0.97	0.98	0.98

TABLE 4.9 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: W(2, 1), CENSORING DISTRIBUTION: E(0.425)
n=10 (approx. 30% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10
0.10	0.20	0.22	0.36	0.56	0.90	1.12	1.38	1.19	1.34	1.00
0.25	0.86	0.90	1.30	1.30	1.47	1.16	1.46	1.13	0.95	1.00
0.50	1.47	1.04	1.96	1.67	1.84	1.17	1.43	1.03	1.10	1.00
0.75	0.48	1.41	2.14	2.21	2.53	2.02	2.04	1.82	1.77	1.00
0.90	0.18	0.35	0.54	0.80	1.01	1.05	1.21	1.32	1.13	1.00
0.95	0.17	0.28	0.39	0.54	0.65	0.69	0.85	0.91	0.98	1.00

TABLE 4.10 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: W(2,1), CENSORING DISTRIBUTION: E(0.865)
n=10 (approx. 50% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10
0.10	0.50	0.32	0.38	0.45	0.58	0.84	1.05	0.84	0.85	1.00
0.25	3.65	2.10	1.93	1.80	1.74	1.83	1.48	1.37	1.27	1.00
0.50	0.96	1.42	2.39	2.39	2.69	1.94	1.77	1.49	1.55	1.00
0.75	0.30	0.69	1.18	1.60	1.98	1.73	1.58	1.60	1.62	1.00
0.90	0.17	0.28	0.41	0.56	0.68	0.70	0.82	0.85	1.13	1.00
0.95	0.21	0.32	0.42	0.54	0.63	0.65	0.75	0.78	1.03	1.00

TABLE 4.11 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: E(1), CENSORING DISTRIBUTION: E(3/7)
n=10 (approx. 30% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10
0.10	0.07	0.13	0.23	0.45	0.53	1.29	2.10	2.44	2.43	1.00
0.25	0.24	0.48	1.01	1.84	0.67	1.01	1.46	1.76	1.53	1.00
0.50	4.26	3.09	2.50	2.32	1.32	1.76	1.56	1.92	1.30	1.00
0.75	1.20	2.64	3.32	2.40	1.94	2.13	1.68	1.99	1.46	1.00
0.90	0.34	0.51	0.74	0.86	0.96	1.13	1.06	1.08	1.32	1.00
0.95	0.33	0.44	0.57	0.64	0.72	0.82	0.87	0.99	1.03	1.00

TABLE 4.12 RATIOS OF MEAN SQUARE ERRORS OF PL QUANTILE ESTIMATION AND GENERALIZED PL QUANTILE ESTIMATION

LIFE DISTRIBUTION: E(1), CENSORING DISTRIBUTION: E(1)
n=10 (approx. 50% CENSORING)

p \ k	1	2	3	4	5	6	7	8	9	10
0.10	0.17	0.20	0.25	0.35	0.37	0.68	1.52	1.15	2.02	1.00
0.25	1.00	1.13	1.53	1.76	0.92	1.08	0.97	1.19	1.48	1.00
0.50	3.82	2.87	4.83	3.32	1.79	1.76	1.13	1.23	1.50	1.00
0.75	0.55	0.91	1.48	1.61	1.65	1.61	1.57	1.28	1.50	1.00
0.90	0.37	0.48	0.61	0.70	0.77	0.82	0.94	1.03	1.01	1.00
0.95	0.47	0.56	0.66	0.73	0.79	0.83	0.92	1.00	0.98	1.00

FIGURE 5.1. GENERALIZED QUANTILE ESTIMATE
n = 15, k = 8

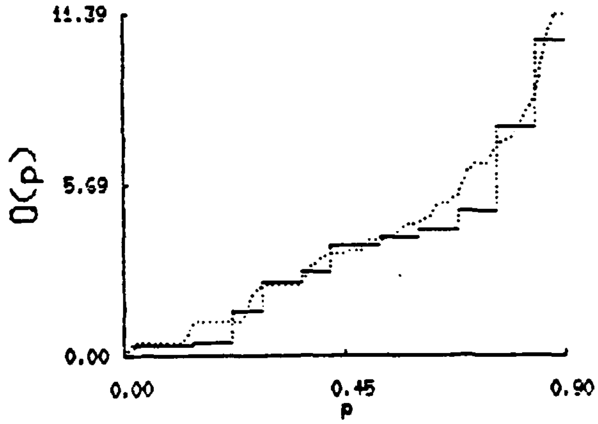


FIGURE 5.2. GENERALIZED QUANTILE ESTIMATE
n = 15, k = 10

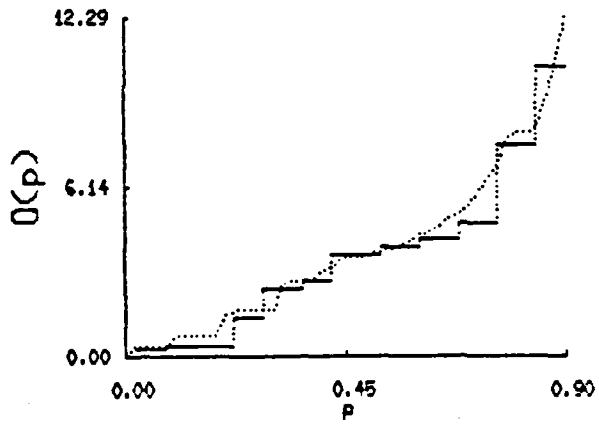
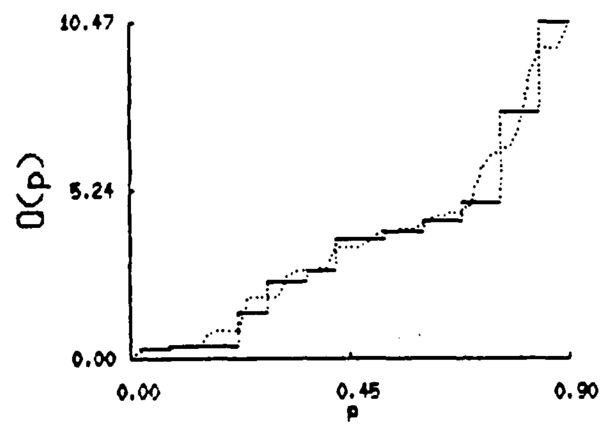


FIGURE 5.3. GENERALIZED QUANTILE ESTIMATE
n = 15, k = 13



ACKNOWLEDGEMENTS

This research was supported by the U. S. Air Force Office of Scientific Research Grant AFOSR-84-0156 and the U. S. Army Research Office grant MIPR ARO 139-85.

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KEY WORDS: Random censoring; Product-Limit quantile estimator; Generalized Product-Limit quantile estimator; Median survival time estimation; Nonparametric quantile estimation.

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